F. Formalism

a category consists of 3 things:

(1) a collection $Ob(\mathcal{C})$ of objects

(2) for each pair $X, Y \in Ob(\mathcal{C})$ a set

$$Mor(X,Y)$$

of morphisms such that $Mor(X,X)$ has a distinguised elt $\mathbb{1}_X$

(3) for each triple $X, Y, Z \in Ob(\mathcal{C})$ a composition

$$\circ : Mor(X,Y) \times Mor(Y,Z) \to Mor(X,Z)$$

satisfying
$$f \circ \mathbb{1} = f$$
$$\mathbb{1} \circ f = f$$
$$(f \circ g) \circ h = f \circ (g \circ h)$$

examples

1) category of topological spaces $\mathcal{T}$

$Ob(\mathcal{T})$ = topological spaces

$Mor(X,Y) = \{$continuous functions $f: X \to Y\}$

composition is composition of functions

2) $\mathcal{CW} \subset \mathcal{T}$ $\quad Ob(\mathcal{CW}) = \{$CW complexes$\}$

Mor and $\circ$ from $\mathcal{T}$

3) category of groups $\mathcal{G}$ and homomorphisms

4) $Ab \subset \mathcal{G}$ abelian groups

5) homotopy category $\mathcal{H}$

$Ob(\mathcal{H}) = \{$topological spaces$\}$

$Mor(X,Y) = \{$homotopy classes of maps $X \to Y\}$

6) category of chain complexes $\mathcal{C}$

$Ob(\mathcal{C}) = \{$chain complexes$\}$

$Mor(X,Y) = \{$chain maps $X \to Y\}$

$\circ$ is composition of chain maps

7) if G a group we have a category $\mathcal{C}_G$ where

$Ob(\mathcal{C}_G) = \{*\}$

$Mor(*,*) = G$

$\circ$ is group multiplication

so morphisms donot have to be maps!

a covariant functor F from the category $\mathcal{C}$ to the category $\mathcal{D}$

assignes to each $X \in Ob(\mathcal{C})$ an object $F(X) \in Ob(\mathcal{D})$

and to each $f \in Mor(X,Y)$ a morphism $F(f) \in Mor(F(X), F(Y))$

s.t. $F(\mathbb{1}_X) = \mathbb{1}_{F(X)}$ and

$F(f \circ g) = F(f) \circ F(g)$

a contravariant functor is the same except if $f \in Mor(X,Y)$ then $F(f) \in Mor(F(Y), F(X))$ and $F(f \circ g) = F(g) \circ F(f)$

examples:

fundamental group; homotopy category of pointed spaces

1) π_1 is a covariant functor from $\mathcal{H}^*$ to $\mathcal{G}$ ← category of groups

topological spaces

2) C_* is a covariant functor from $\mathcal{T}$ to $\mathcal{C}$ ← chain complexes

singular chain complex

3) H_* is a covariant functor from $\mathcal{C}$ to $\{\text{graded abelian groups}\} = \mathcal{GR}$

4) $H \circ C$ is a functor from $\mathcal{T}$ to $\mathcal{GR}$

and induces one from $\mathcal{H}$ to $\mathcal{GR}$

5) for fixed n $H_n \circ \mathcal{C}$ is a functor $\mathcal{H}$ to $\mathcal{Ab}$ ← abelian groups

6) $\mathcal{V}$ = category of vector spaces and linear maps

duality $*$ is a contravariant functor $\mathcal{V}$ to $\mathcal{V}$

a natural transform T between functors $F, G : \mathcal{C} \to \mathcal{D}$ is an assignment of a morphism $T_X : F(X) \to G(X)$ $\forall X \in Ob(\mathcal{C})$

such that for each morphism $f \in Mor(X,Y)$ in $\mathcal{C}$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"] \arrow[dr, phantom, "\circ"] & F(Y) \arrow[d, "T_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

examples:

1) let $\mathcal{PT}$ be category of pairs (X,A) of topological spaces

for each n, H_n is a functor from $\mathcal{PT}$ to $\mathcal{A}$

given $(X,A) \in Ob(\mathcal{PT})$ let

$$\partial_n : H_n(X,A) \to H_{n-1}(A) = H_{n-1}(A,\emptyset)$$

be the map in the long exact sequence of a pair

(and if $B \neq A$ then define $\partial_n : H_n(X,A) \to H_n(B,C)$ to be 0)

then ∂_n is a natural transform

2) If $f: G_1 \to G_2$ a homeomorphism of abelian groups

then we get an induced map $H_*(X;G_1) \to H_*(X;G_2)$

that behaves "naturally" with respect to maps $X \to Y$

this gives a natural transformation $H_*(\cdot\,; G_1)$ to $H_*(\cdot\,; G_2)$

a (generalized) homology theory is a set of functors

$$h_n : \mathcal{PT} \longrightarrow \mathcal{A}$$

together with natural transforms $\partial_n : h_n(X,A) \to h_{n-1}(A,\emptyset)$

satisfying 1) (Homotopy) if $f,g : (X,A) \to (Y,B)$ are homotopic as maps of pairs, then $h_n(f) = h_n(g)$

2) (Exactness) $\forall$ pairs (X,A), $i: A \to X$, $j: (X,\emptyset) \to (X,A)$

the sequence

$$h_n(A) \xrightarrow{h_n(i)} h_n(X) \xrightarrow{h_n(j)} h_n(X,A) \xrightarrow{\partial_n} h_{n-1}(A)$$

is exact $\forall n$

3) (Excision) if $Z \subset \bar{Z} \subset \text{int} A \subset A \subset X$ then the inclusion map $i:(X-Z, A-Z) \to (X,A)$ induces an isomorphism $h_n(i): h_n(X-Z, A-Z) \to h_n(X,A) \quad \forall n$

4) (Additivity) if (X,A) is a disjoint union of pairs (X_λ, A_λ), $\lambda \in I$, then the inclusion maps $i_\lambda: (X_\lambda, A_\lambda) \to (X,A)$ induce an isomorphism $\oplus_\lambda (i_\lambda)_*: \oplus_\lambda h_n(X_\lambda, A_\lambda) \to h_n(X,A)$

Th^{m} 28 (Eilenberg-Steenrod):

If $\{h_n\}$ and $\{\partial_n\}$ is a generalized homology theory that satisfies

(Dimension) $h_n(pt) = \begin{cases} G & n=0 \\ 0 & n\neq 0 \end{cases}$ some abelian group G

singular homology

then for any CW pair (X,A), $h_n(X,A) \cong H_n(X,A;G)$

Idea of proof: look back at section D and see you can compute $h_n(X,A)$ for any CW-complex based only on the axioms so any 2 such functors will need to give the same answer for (X,A)

G. Geometric Interpretation of homology

the elements of $H_n(X)$ might be, at the moment, somewhat mysterious

here is something that seems more concrete

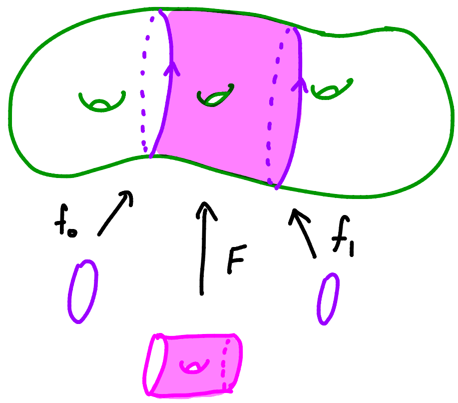
let M^k, N^k be 2 smooth oriented manifolds

call 2 maps $f_0: M^k \to X$ and $f_1: N^k \to X$ cobordant if $\exists$ a smooth oriented manifold W^{k+1} and a map $F: W \to X$ such that

reverse orientation

$\partial W = -M \cup N$,

$F|_M = f_0$, and $F|_N = f_1$



let $\Omega_k(X)$ = cobordism classes of maps of k-manifolds into X
group structure is disjoint union

this is nice since we can think of

$\Omega_1(X)$ as union of S^1's in X (modulo surfaces)

$\Omega_2(X)$ as union of surfaces in X (modulo 3-manifolds)

$\vdots$

however geometrically appealing this might be it is actually very complicated

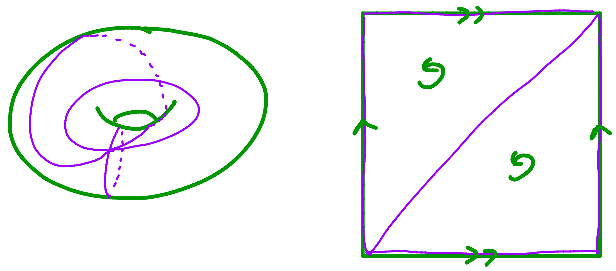
$$\Omega_k(pt) = \begin{cases} \mathbb{Z} & k=0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 3 \\ \mathbb{Z} & 4 \\ \mathbb{Z}/2 & 5 \\ 0 & 6 \\ 0 & 7 \\ \mathbb{Z}\oplus\mathbb{Z} & 8 \\ \mathbb{Z}/2\oplus\mathbb{Z}/2 & 9 \\ \mathbb{Z}/2 & 10 \\ \mathbb{Z}/2 & 11 \\ \neq 0 & n\geq 8 \end{cases}$$

where as $H_k(pt) = \begin{cases} \mathbb{Z} & k=0 \\ 0 & k\neq 0 \end{cases}$

<u>Fact:</u> Ω_k is a generalized homology theory

every smooth manifold M^n has a <u>triangulation</u>, that is a union of embedded n-simplicies with disjoint interiors s.t. each $(n-1)$-simplex is the face of exactly one n-simplex (if $\partial M \neq \emptyset$) or two n-simplicies (and the $(n-1)$-simplex is oriented oppositely by two n-simplicies)

example:



lemma 29:

1) if M is an oriented, smooth, compact n-manifold then any map $f: M \to X$ defines an n-chain in $C_n(X)$
2) if M does not have boundary then f defines an n-cycle (and hence an element $[f(M)]$ in $H_n(X)$)
3) if M does not have boundary and f and g are homotopic, then $[f(M)] = [g(M)] \in H_n(X)$
4) if W is a compact $(n+1)$-manifold with $\partial W = -M_0 \amalg M_1$ and $F: W \to X$ is a map, then $[F|_{M_0}] = [F|_{M_1}]$ in $H_n(X)$

Proof idea: triangulate M with simplicies $\sigma_1 \dots \sigma_k$
then $f \circ \sigma_1$ are singular n-simplicies
can assemble them so all statements true $\blacksquare$

Remark: So we get a map $\Omega_k(X) \to H_k(X)$

1) map surjective for $k \leq 6$ (so any homology class is "realized" by a manifold!)

2) map is an isomorphism for $k \leq 3$

3) for $k \geq 7$ $\exists$ homology classes that can't be realized by a manifold but $\forall\, \alpha \in H_k(X)$ some multiple $n\alpha$ can be realized.